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EXACT SOLUTIONS OF THE BOLTZMANN–MAXWELL KINETIC EQUATION†

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Exact solutions of the following topics are reviewed: locally Maxwell solutions, spatially uniform relaxation of a binary mixture of gases (moment solutions and a generalization of the Bobylev–Krook–Wu solutions), homoenergetic affine flows (the Galkin–Truesdell class of exact solutions), spherical outflow-inflow (the Nikol'skii transformation), dominant solutions and power solutions. New results are obtained on the second and third topics. Particular attention is devoted to the qualitative properties of the solutions, which are of fairly general interest. We mean by an exact solution a solution in explicit form, i.e. in terms of elementary or transcendental functions, of the non-linear Boltzmann–Maxwell kinetic equation (the distribution function) or the Maxwell transport equations (the moments of the distribution functions). The latter are also called moment solutions of the kinetic equation. The molecules of the gas are mainly assumed to be Maxwellian, when the coefficient of viscosity and thermal conductivity depend linearly on the temperature. © 2004 Elsevier Ltd. All rights reserved.

In this paper we attempt to re-established historical correctness [1]. As is well known [1–3], Maxwell introduced the fundamental statistical idea of a distribution function, and laid the foundations of statistical physics and physical kinetics. When deriving the kinetic moment equations (the Maxwell transport equations) he formulated all that was necessary in order to write the kinetic equation for the distribution function, which was also done by Boltzmann. Hence, in publications at the beginning of the twentieth century the kinetic equation was named after Boltzmann and Maxwell [2], and not just Boltzmann [4–6].

In view of the complexity of the kinetic equation, the exact solutions describe extremely degenerate processes. The computer revolution enabled numerical solutions of a large number of problems to be obtained. Nevertheless, considerable attention has been devoted to exact solutions in kinetic theory [1, 4–6].

Because of limitations of space we do not consider papers in which simplified representations of the collision operator (linearized or model operators) are employed or assume that the external forces depend on the velocities of the molecules. Preference is given to concluding papers published in easily accessible journals. Foremost attention is devoted to new results and results which have not been given sufficient attention in existing textbooks [1, 4–6].

1. INITIAL RELATIONS

We will write the system of Maxwell–Boltzmann kinetic equations in the variables $\mathbf{C}_i$, $\mathbf{r}$, t, where the peculiar velocity of a particle of the ith component of a mixture of monatomic gases $\mathbf{C}_i = \boldsymbol{\xi}_i - \mathbf{u}$, $\boldsymbol{\xi}_i$ is the absolute velocity of a particle and $\mathbf{u}$ is the mean-mass velocity

$$\mathbf{u} = \frac{1}{\rho}\sum m_i \int \boldsymbol{\xi}_i f_i(\boldsymbol{\xi}_i, \mathbf{r}, t) d\boldsymbol{\xi}_i$$

Here and below the summation sign denotes summation over all the values of $i = 1, 2, \ldots, N$, where N is the number of components of the mixture.

This system has the form

$$\begin{gathered}\frac{Df_i}{Dt} + C_{i\beta}\frac{\partial f_i}{\partial r_\beta} + \left(F_{i\beta} - \frac{Du_\beta}{Dt}\right)\frac{\partial f_i}{\partial C_{i\beta}} - \frac{\partial f_i}{\partial C_{i\beta}} C_{i\gamma}\frac{\partial u_\beta}{\partial r_\gamma} = J_i \\ \frac{D}{Dt} = \frac{\partial}{\partial t} + u_\beta \frac{\partial}{\partial r_\beta}, \quad f_i = f_i(\mathbf{C}_i, \mathbf{r}, t), \quad \mathbf{F}_i = \mathbf{F}_i(\mathbf{r}, t)\end{gathered} \tag{1.1}$$

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Here f_i is the distribution function of the peculiar velocities $\mathbf{C}_i$ of the monatomic particles of the ith component, $\mathbf{F}_i$ is the external force, referred to the particle mass m_i, J_i is the collision operator and r_α are the components of the radius vector ($r_1 = x, r_2 = y$ and $r_3 = z$); summation over repeated subscripts $\alpha, \beta, \gamma = 1, 2, 3$ is used and t is the time.

The central moments of the distribution function are introduced by the formula

$$M^{(n)}_{i\{\alpha\}} = m_i\int \mathbf{C}_i^{(n)} f_i d\mathbf{C}_i \equiv m_i\int C_{i\alpha_1}C_{i\alpha_2}\ldots C_{i\alpha_n} f_i d\mathbf{C}_i, \quad \{\alpha\} = \alpha_1\alpha_2\ldots\alpha_n, \quad n\ge 0 \qquad (1.2)$$

In particular, the mass density and the diffusion velocity of the ith component $\rho_i = M_i^{(0)}$, $\mathbf{V}_i = M_i^{(1)}/\rho_i$, and the temperature, the components of the pressure tensor and the vector of the heat flux respectively are equal to

$$(T, \tau_{\alpha\beta}, q_\alpha) = \sum\left\{\frac{1}{3nk}M^{(2)}_{i\alpha\alpha}, M^{(2)}_{i\alpha\beta}, \frac{1}{2}M^{(3)}_{i\alpha\beta\beta}\right\}, \quad n = \sum n_i, \quad n_i = \frac{\rho_i}{m_i}$$

where

$$\tau_{\alpha\beta} = p_{\alpha\beta} + \delta_{\alpha\beta}p, \quad p = nkT, \quad p_{\alpha\alpha} = 0$$

Here $\delta_{\alpha\beta}$ and the stresses $p_{\alpha\beta}$ are the components of the identity and non-divergent tensors respectively, p is the hydrostatic pressure (henceforth called simply the pressure), and k is Boltzmann's constant. In hydrodynamics the quantities $\sigma_{\alpha\beta} = -p_{\alpha\beta}$ are called the viscous stresses.

The Maxwell transport equations (the kinetic moment equations)

$$\begin{aligned}&\frac{D}{Dt}M^{(n)}_{i\{\alpha\}} + \frac{\partial}{\partial r_\beta}M^{(n+1)}_{i\{\alpha\}\beta} - \left(F_{i\beta} - \frac{Du_\beta}{Dt}\right)m_i\int\frac{\partial \mathbf{C}_i^{(n)}}{\partial C_{i\beta}} f_i d\mathbf{C}_i + \\ &+ \frac{\partial u_\beta}{\partial r_\gamma} m_i \int \frac{\partial}{\partial C_{i\beta}}(C_{i\gamma}\mathbf{C}_i^{(n)}) f_i d\mathbf{C}_i = R^{(n)}_{i\{\alpha\}} \equiv m_i\int \mathbf{C}_i^{(n)} J_i d\mathbf{C}_i\end{aligned} \qquad (1.3)$$

follow from relations (1.1) and (1.2). Hence we have the conservation equations

$$\frac{\partial \rho_i}{\partial t} + \nabla\rho_i(\mathbf{u} + \mathbf{V}_i) = 0, \quad \nabla\mathbf{A} = \frac{\partial A_\alpha}{\partial r_\alpha} \qquad (1.4)$$

$$\frac{\partial \rho}{\partial t} + \nabla\rho\mathbf{u} = 0 \qquad (1.5)$$

$$\rho\frac{Du_\alpha}{Dt} + \frac{\partial p}{\partial r_\alpha} + \frac{\partial p_{\alpha\beta}}{\partial r_\beta} - \sum\rho_i F_{i\alpha} = 0 \qquad (1.6)$$

$$\frac{3}{2}nk\frac{DT}{Dt} + \frac{\partial q_\alpha}{\partial r_\alpha} + (p\delta_{\alpha\beta} + p_{\alpha\beta})\frac{\partial u_\alpha}{\partial r_\beta} - \frac{3}{2}kT\frac{\partial}{\partial r_\alpha}\sum n_i V_{i\alpha} - \sum\rho_i F_{i\alpha}V_{i\alpha} = 0 \qquad (1.7)$$

Equations (1.3) are an infinite system: the equation for $M^{(n)}_{i\{\alpha\}}$ contains spatial derivatives of $M^{(n+1)}_{i\{\alpha\}\beta}$; with the exception of cases (1.4)–(1.7), the right-hand sides of the transport equations $R^{(n)}_{i\{\alpha\}} \neq 0$ and depend on moments of a different order.

The case of Maxwell molecules is a special one (for them the intermolecular force of interaction of the particles i and j is equal to $\kappa_{ij}r^{-5}$, where $i, j = 1, 2, \ldots, N$ and r is the distance between them; the transport coefficients are proportional to T), when $R^{(n)}_{i\{\alpha\}}$ are expressed explicitly in terms of moments of order $m \le n$. The discovery of these molecules is due to Maxwell's interest in the transport equations. In this case, using system of equations (1.3) it is possible to determine certain moments of the distribution function, without knowing the function itself (Sections 3, 5 and 7), i.e. it is possible to obtain moment solutions of the Boltzmann–Maxwell kinetic equation.

Below, the molecules will be mainly assumed to be Maxwellian, unless otherwise stated.

The most general results in calculating $R^{(n)}_{i\{\alpha\}}$ are obtained for a simple gas (i.e. for a single-component monatomic gas, when the subscript i is omitted). Expressions for $R^{(2)}$ and $R^{(3)}$ are given in [1.6], and an expression for $R^{(4)}$ is given in [1]. Formulae have also been obtained for $R^{(5)}$ and $R^{(6)}$ in [7]. An effective method of calculating $R^{(n)}$ is proposed in [8].

In particular, the following system of equations holds for the stresses $p_{\alpha\beta}$ of a simple gas [1, 6]

$$\frac{\partial p_{\alpha\beta}}{\partial t}+\frac{\partial}{\partial r_\gamma}(u_\gamma p_{\alpha\beta})+\frac{\partial Q_{\alpha\beta\gamma}}{\partial r_\gamma}-\frac{2}{3}\delta_{\alpha\beta}\nabla\mathbf{q}+2\left\langle p_{\alpha\gamma}\frac{\partial u_\beta}{\partial r_\gamma}\right\rangle+2p\left\langle\frac{\partial u_\alpha}{\partial r_\beta}\right\rangle=-\frac{p}{\mu}p_{\alpha\beta} \tag{1.8}$$

Here $\mu = \mu_0 T$ is the coefficient of dynamic viscosity ($\mu_0 = \text{const}$), and the operation $\langle\ \rangle$ is introduced by the formula

$$\langle A_{\alpha\beta}\rangle = \frac{1}{2}(A_{\alpha\beta}+A_{\beta\alpha})-\frac{1}{3}\delta_{\alpha\beta}A_{\gamma\gamma} \tag{1.9}$$

The third-order moments are

$$Q_{\alpha\beta\gamma} = m\int C_\alpha C_\beta C_\gamma f d\mathbf{C},\quad q_\alpha = \frac{1}{2}Q_{\alpha\beta\beta} \tag{1.10}$$

The system of equations for these is also derived in [1, 6].

A more general system of transport equations is obtained if we introduced the mean velocities $\mathbf{u}_i$ and the mean temperatures T_i, the stress $p_{i\alpha\beta}$, and the heat fluxes $\mathbf{q}_i$ of the components of a mixture of gases by the formulae

$$[\mathbf{u}_i, T_i, p_{i\alpha\beta}, \mathbf{q}_i] = \int\left[\frac{\boldsymbol{\xi}_i}{n_i}, \frac{m_i}{3n_ik}C_i^{*2}, m_i\left(C^*_{i\alpha}C^*_{i\beta}-\frac{1}{3}\delta_{\alpha\beta}C_i^{*2}\right), \frac{m_i}{2}\mathbf{C}_i^*C_i^{*2}\right]f_i(\boldsymbol{\xi}_i, \mathbf{r}, t)d\boldsymbol{\xi}_i \tag{1.11}$$

These transport equations are obtained from Eqs (1.1)–(1.3) by replacing $\mathbf{u}$ and $\mathbf{C}_i$ by $\mathbf{u}_i$ and $\mathbf{C}^*{}_i = \boldsymbol{\xi}_i - \mathbf{u}_i$, while retaining the remaining notation. An analysis which we have carried out showed that, with the least errors, the moments $R^{(n)}_{i\{\alpha\}}$ ($n = 1, 2, 3, 4$) corresponding to (1.11) are obtained in [9]: it is only necessary to add the factor γ in front of q^*_{bn} in formula (6) [9]. In formula (10) from [10] we must add terms proportional to WT_i, and in F_{11} we must replace $4m_{10}m_{20}$ by $1–4m_{10}m_{20}$.

In the case of a binary mixture we have the relations

$$T = x_1T_1+x_2T_2+\frac{\rho_1\rho_2}{3nk\rho}\Delta^2,\quad \boldsymbol{\Delta} = \mathbf{u}_1-\mathbf{u}_2,\quad x_i = \frac{n_i}{n} \tag{1.12}$$

2. LOCALLY MAXWELL'S SOLUTIONS

The locally Maxwell distribution function of gas molecules

$$f^M = n(2\pi RT)^{-3/2}\exp\left[-\frac{(\boldsymbol{\xi}-\mathbf{u})^2}{2RT}\right] \tag{2.1}$$

causes the collision integral to vanish identically, and the dependence on n, T, $\mathbf{u}$ on $\mathbf{r}$, t is determined by the left (convective) part of the kinetic equation, and these gas-dynamic variables naturally satisfy Euler's equations. This also occurs for a gas mixture if all the Maxwellians f_i^M are determined for the same T and $\mathbf{u}$. In the results described earlier we need only replace n and $R = k/m$ by n_i and R_i.

The most complete analysis for the case of a simple gas when there are no external forces is given in [11] (external forces are taken into account, for example, in the monographs [4, 6]). We will use the notation from [11], replacing β by β_g; we have

$$\begin{gathered}T = -\frac{1}{2R\gamma_4},\quad \gamma_4 = a_4+\beta_g t+\alpha t^2,\quad \boldsymbol{\gamma} = \mathbf{a}-\mathbf{k}t-(2\alpha t+\beta_g)\mathbf{r}+\boldsymbol{\Omega}\times\mathbf{r}\\ \mathbf{u} = -\frac{\boldsymbol{\gamma}}{2\gamma_4},\quad \ln[n(2\pi RT)^{-3/2}] = \Phi,\quad \Phi = a_0+\mathbf{k}\cdot\mathbf{r}+\alpha r^2-\frac{\gamma^2}{4\gamma_4}\end{gathered} \tag{2.2}$$

The coefficients of the powers of t and r_α are arbitrary constants. The three components of the asymmetric tensor $\Omega_{\alpha\beta}$ are components of the vector $\mathbf{\Omega}$.

The most important particular solutions were obtained, as is well known, by Maxwell. We obtain the Maxwell equilibrium distribution function when only the constants a_0, $\mathbf{a}$, $a_4 < 0$ are non-zero. If only a_0, Ω and $a_4 < 0$ are non-zero, then

$$\begin{gathered} T = -\frac{1}{2Ra_4} = \text{const}, \quad \mathbf{u} = -\frac{1}{2a_4}\mathbf{\Omega}\times\mathbf{r} \\ n = (2\pi RT)^{3/2}\exp\left[a_0 - \frac{1}{4a_4}(\mathbf{\Omega}\times\mathbf{r})^2\right] \end{gathered} \tag{2.3}$$

The solution (2.1), (2.3) describes the rotation of the gas as a rigid body. The temperature is uniform, and the density and pressure increase exponentially with the radius (measured from the axis of rotation) and are independent of t.

In the general case the temperature $T = T(t)$, and the gas motion is the superposition of a uniform flow, spherical outflow-inflow and rotation of the gas as a rigid body, and the density and pressure are functions of t and $\mathbf{r}$. If $\alpha \neq 0$, but $\mathbf{a}$, β_g, $\mathbf{k}$ and $\mathbf{\Omega}$ are equal to zero, then

$$\begin{gathered} T = -\frac{1}{2R(a_4 + \alpha t^2)}, \quad \mathbf{u} = \frac{\alpha t\mathbf{r}}{a_4 + \alpha t^2} \\ n = (2\pi RT)^{3/2}\exp\left[a_0 + \frac{\alpha a_4 r^2}{a_4 + \alpha t^2}\right] \end{gathered} \tag{2.4}$$

Henceforth we will establish the initial conditions when $t = 0$ (and not $t = t_0$). From the condition $T > 0$ it follows that $a_4 < 0$. If $\alpha > 0$, a solution exists in a finite range of values of t. When $\alpha a_4 > 0$ the total mass is finite [11]

$$M = \int \rho d\mathbf{r} = \pi^3 m(\alpha a_4)^{-3/2}\exp a_0$$

We will consider a special case that will be important later, namely, $\alpha \neq 0$ and $\mathbf{\Omega} = 0$, and the density (and pressure) depend only on t. Substituting expressions (2.2) for γ_4 and γ into the relation for Φ and equating the coefficients of $\mathbf{r}$ and r^2 to zero, then when $t = 0$ and $t \neq 0$, we obtain

$$\begin{gathered} a_4 = \frac{\beta_g^2}{4\alpha}, \quad \mathbf{a} = -\frac{\beta_g}{2\alpha}\mathbf{k}, \quad \gamma_4 = \frac{1}{4\alpha}(\beta_g + 2\alpha t)^2 \\ \mathbf{u} = \frac{\mathbf{k} + 2\alpha\mathbf{r}}{\beta_g + 2\alpha t}, \quad \Phi = a_0 - \frac{k^2}{4\alpha}, \quad T(0) = -\frac{2\alpha}{R\beta_g^2} \end{gathered}$$

The finiteness of T requires that $\beta_g \neq 0$. Putting

$$2\alpha\mathbf{r}' = \mathbf{k} + 2\alpha\mathbf{r}, \quad t^* = t/c, \quad c = \beta_g/(2\alpha)$$

and omitting the prime, we finally obtain

$$\rho = \frac{\rho(0)}{(1+t^*)^3}, \quad \mathbf{u} = \frac{c^{-1}\mathbf{r}}{1+t^*}, \quad T = \frac{T(0)}{(1+t^*)^2} \tag{2.5}$$

$$f^M = n(0)(2\pi RT(0))^{-3/2}\exp\left(-\frac{C^{*2}}{2RT(0)}\right), \quad \mathbf{C}^* = (1+t^*)(\boldsymbol{\xi} - \mathbf{u}) \tag{2.6}$$

If $\beta_g < 0$, then $c > 0$ and we have spherical outflow of the gas, and when $\beta_g > 0$ we have spherical inflow ($c < 0$) for $t \in [0, 1/|c|)$. In the first case (ρ, $\mathbf{u}$, T, f) $\to 0$ as $t^* \to \infty$, and in the second (ρ, $\mathbf{u}$, T) $\to \infty$, and f tends to a constant as $t^* \to -1$ (the initial values, $\mathbf{r}$ and $\boldsymbol{\xi}$ are fixed).

Consideration of the flow (2.5) and (2.6) will be continued in Section 6.

Everywhere $t^* = t/t_0$ is dimensionless time, but the quantity t_0 is given by different expressions.

3. SPATIALLY UNIFORM RELAXATION OF A BINARY MIXTURE (THE MOMENT SOLUTIONS)

Spatially uniform relaxation is described by the solution of the Cauchy problem for the following system of kinetic equations

$$\frac{\partial f_i}{\partial t} = J_i, \quad f_i(0, \xi_i) = \omega_i(\xi_i) \tag{3.1}$$

when there are no external forces. Here ω_i is a specified function of the particle velocity ξ_i, $i = 1, 2$. Problem (3.1) is almost always solved numerically.

In this case, the transport equations (1.3) take the form

$$\frac{\partial}{\partial t} M^{(n)}_{i\{\alpha\}} = R^{(n)}_{i\{\alpha\}} \tag{3.2}$$

When $t = 0$ the initial values of $M^{(n)}_{i\{\alpha\}}(0)$, which satisfy the requirement that the distribution functions must be positive, are specified. The infinite chain of moment equations (3.2) in the case of Maxwell molecules is expanded in a series of successively solved closed systems of equations [6], since in this case the right-hand sides $R^{(n)}_{i\{\alpha\}}$ are expressed in terms of the moments $M^{(m)}$, $m \le n$. For example, we have from system (1.8) $\partial p_{\alpha\beta}/\partial t = -p_{\alpha\beta}/\tau$, whence $p_{\alpha\beta}(t) = p_{\alpha\beta}(0)\exp(-t/\tau)$, where the translational relaxation time $\tau = \mu/p$ in this case is constant. Interesting results were obtained in [10, 12–14] for the relaxation of a binary mixture of gases with sharply differing masses of the molecules, when the parameter

$$\varepsilon = \sqrt{m_1/m_2} \tag{3.3}$$

is much less than unity. Then, when $\mathbf{u}_1 \ne \mathbf{u}_2$, $T_1 \ne T_2$, the relaxation process is split into qualitatively different stages (m_1 and m_2 are the masses of the particles of the "light" and "heavy" components of the mixture). The peculiarity of this process is described in many handbooks [15]. However, the existing analysis of the relaxation process, as it is clear now, has been carried out with simplifying assumptions. As a result, important properties of the process have been overlooked.

Considered a binary mixture of monatomic gases, consisting of Maxwell molecules, putting

$$\mathbf{\Delta} = \mathbf{u}_1 - \mathbf{u}_2, \quad \Delta_T = T_1 - T_2 \tag{3.4}$$

The number densities n_i and the temperature T are constant by virtue of relations (1.4) and (1.7). We will use definitions (1.11). To determine the functions (3.4) we have the equations

$$\begin{gathered}\frac{\partial \mathbf{\Delta}}{\partial t} = -A\rho\mathbf{\Delta}, \quad \frac{\partial \Delta_T}{\partial t} = -2An\mu_m\Delta_T - \frac{2}{3k}(\rho_2 - \rho_1)A\mu_m\Delta^2 \\ A = \frac{2\pi A_1}{m_0}\left(\frac{\kappa_{12}}{\mu_m}\right)^{1/2}, \quad m_0 = m_1 + m_2, \quad \mu_m = \frac{m_1 m_2}{m_0}\end{gathered} \tag{3.5}$$

We will assume that the intermolecular forces are equal to $\kappa_{ij}r^{-5}$, where $i, j = 1, 2$ and r is the distance between particles.

We have from the first equation of (3.5)

$$\mathbf{\Delta}(t^*) = \mathbf{\Delta}(0)\exp(-t^*), \quad t^* = \frac{t}{\tau_u}, \quad \tau_u = \frac{1}{A\rho} \tag{3.6}$$

We will write the solution of the second equation of (3.5) as

$$\frac{\Delta_T(t^*)}{T} = \left[\frac{\Delta_T(0)}{T} - h\Delta^2(0)\right]\exp\left(-\frac{t^*}{\tau_T^*}\right) + h\Delta^2(0)\exp(-2t^*) \tag{3.7}$$

We have used the following notation in solution (3.7)

$$\tau_T^* = \frac{\tau_T}{\tau_u} = \frac{\rho}{2n\mu_m} \approx \frac{x_2}{2\varepsilon^2}, \quad h = \frac{A\mu_m}{3kT}(\rho_2 - \rho_1)\tau_u \approx \frac{m_1}{3kT} \tag{3.8}$$

The quantity τ_T^* is equal to the ratio of the relaxation times of the temperature and velocity differences.

On the right-hand sides of formula (3.8) we have introduced approximate values, obtained for $\varepsilon = \sqrt{m_1/m_2} \to 0$ and fixed remaining parameters of the gases and of the initial values (in particular, x_i). Under these conditions the relaxation times differ sharply: $\tau_T^* \gg 1$.

Previously in [13, 14] the quantity $h\Delta^2(0)$ was in fact assumed to be negligibly small. If, however, $\Delta_T(0)/T \sim h\Delta^2(0) \sim 1$, it cannot be neglected and when $t^* \sim \tau_T^*$, $\varepsilon \ll 1$, taking into account the fact that $\tau_T \gg \tau_u$, we obtain

$$\Delta_T(t^*) \approx [\Delta_T(0) - Th\Delta^2(0)]\exp\left(-\frac{t^*}{\tau_T^*}\right) \tag{3.9}$$

This result is of considerable importance when deriving the equations of two-fluid gas dynamics [16].

Usually it is implicitly assumed that the relaxation has a purely exponential form. However, when the non-uniform parts of the transport equations are taken into account the form of the relaxation turns out to be more complex. For example, for certain values of the parameters of the problem, one of the eigenvalues λ of the equations for the stresses is equal to the coefficient in the exponential function occurring in the non-uniform part, and a term appears in the solution proportional to $t\exp(-\lambda t)$.

We will consider the general problem of the relaxation of the stresses $p_{i\alpha\beta}$. We will introduce the following notation.

$$\pi_i = \frac{p_{i\alpha\beta}}{p}, \quad U = 2h\left[\Delta_\alpha(0)\Delta_\beta(0) - \frac{1}{3}\delta_{\alpha\beta}\Delta^2(0)\right] \tag{3.10}$$

Then the system of equations for the stresses [9, 10] takes the form

$$\begin{gathered}
\frac{d\pi_i}{dt^*} = \Gamma_{i1}\pi_1 + \Gamma_{i2}\pi_2 + \Gamma_{i3}U\exp(-2t^*) \\
\Gamma_{11} \equiv -\Gamma_{11}^* = -\gamma_1\frac{\sqrt{1+\varepsilon^2}}{n^* + \varepsilon^2} - 2n^*\alpha(B + \varepsilon^2), \quad \alpha = \frac{1}{(n^* + \varepsilon^2)(1 + \varepsilon^2)} \\
\Gamma_{12} = \alpha L\varepsilon^2, \quad \Gamma_{13} = \frac{3Ln^*}{2(1 + n^*)(n^* - \varepsilon^2)}, \quad \Gamma_{21} = n^*\Gamma_{12} \\
\Gamma_{22} \equiv -\Gamma_{22}^* = -\gamma_2 n^*\varepsilon\frac{\sqrt{1+\varepsilon^2}}{n^* + \varepsilon^2} - 2\alpha\varepsilon^2(1 + B\varepsilon^2), \quad \Gamma_{23} = \Gamma_{13}\varepsilon^2
\end{gathered} \tag{3.11}$$

Here

$$n^* = \frac{n_2}{n_1}, \quad B = \frac{3A_2}{4A_1}, \quad L = 2(1 - B), \quad \gamma_i = B\left(\frac{2\kappa_{ii}}{\kappa_{12}}\right)^{1/2} \tag{3.12}$$

According to well-known results [2], $A_1 = 0.422$ and $A_2 = 0.436$. The eigenvalues of system (3.11) are as follows:

$$\lambda_{1,2} = \frac{1}{2}\{\Gamma_{11}^* + \Gamma_{22}^* \pm [(\Gamma_{11}^* - \Gamma_{22}^*)^2 + 4\Gamma_{12}\Gamma_{21}]^{1/2}\} \tag{3.13}$$

The solution will not be purely exponential if, for example, $\lambda_1 = 2$ (it can be shown that this equality is satisfied for different values of the parameters of the problem). When $\varepsilon \to 0$ and for fixed values of the remaining parameters of the problem we have

$$\lambda_1 \to \lambda_1^0 = 2B + \gamma_1/n^*, \quad \lambda_2 \to \lambda_2^0 = \varepsilon\gamma_2 \tag{3.14}$$

Limiting values are denoted by the zero superscript. The equality $\lambda_1^0 = 2$ arises under the completely realistic condition $\gamma_1 \approx 0.45n^*$. The parameter ε is defined by formula (3.3).

We will write the solution of system (3.11) in the form

$$\pi_i = \left(\frac{C_{i1}^{(1)}}{\lambda_1 - \lambda_2} + C_{i1}^{(2)}U\right)\exp(-\lambda_1 t^*) + \left(\frac{C_{i2}^{(1)}}{\lambda_1 - \lambda_2} + C_{i2}^{(2)}U\right)\exp(-\lambda_2 t^*) + C_{i3}U\exp(-2t^*) \tag{3.15}$$

Suppose the initial values $\pi_i(0) = \pi_i^0$. We will introduce the following notation

$$\alpha^{(1)} = \Gamma_{22}^* + \Gamma_{12}\varepsilon^2, \quad \alpha^{(2)} = \Gamma_{21} + \Gamma_{11}^*\varepsilon^2, \quad \alpha^{(3)} = \frac{\Gamma_{13}}{\lambda_1 - \lambda_2} \tag{3.16}$$

Then

$$\begin{aligned} (C_{13}; C_{23}) &= \frac{\Gamma_{13}}{(2-\lambda_1)(2-\lambda_2)}(\alpha^{(1)} - 2; \alpha^{(2)} - 2\varepsilon^2) \\ C_{11}^{(1)} &= \pi_1^{(0)}(\Gamma_{11}^* - \lambda_2) - \pi_2^0\Gamma_{12}, \quad C_{12}^{(1)} = \pi_1^0(\lambda_1 - \Gamma_{11}^*) + \pi_2^0\Gamma_{12} \\ C_{21}^{(1)} &= (\Gamma_{11}^* - \lambda_1)\left[\pi_1^0\frac{\Gamma_{11}^* - \lambda_2}{\Gamma_{12}} - \pi_2^0\right], \quad C_{22}^{(1)} = (\Gamma_{11}^* - \lambda_2)\left[\pi_1^0\frac{\lambda_2 - \Gamma_{11}^*}{\Gamma_{12}} + \pi_2^0\right] \end{aligned} \tag{3.17}$$

The coefficients C_{i3} naturally vanish when $\lambda_{1,2} = 2$. We will write the coefficients $C^{(2)}$ in two forms, convenient for transferring to resonance cases,

$$\begin{aligned} C_{11}^{(2)} &= -\frac{\alpha^{(3)}}{2-\lambda_1}(\alpha^{(1)} - \lambda_1) = -C_{13} - \frac{\alpha^{(3)}}{2-\lambda_2}(\alpha^{(1)} - \lambda_2) \\ C_{12}^{(2)} &= -C_{13} + \frac{\alpha^{(3)}}{2-\lambda_1}(\alpha^{(1)} - \lambda_1) = \frac{\alpha^{(3)}}{2-\lambda_2}(\alpha^{(1)} - \lambda_2) \\ C_{21}^{(2)} &= -\frac{\alpha^{(3)}}{2-\lambda_1}(\alpha^{(2)} - \varepsilon^2\lambda_1) = -C_{23} - \frac{\Gamma_{13}}{2-\lambda_2}(\alpha^{(2)} - 2\varepsilon^2) - \alpha^{(3)}\varepsilon^2 \\ C_{22}^{(2)} &= -C_{23} + \frac{\alpha^{(3)}}{2-\lambda_1}(\alpha^{(2)} - 2\varepsilon^2) + \alpha^{(3)}\varepsilon^2 = \frac{\alpha^{(3)}}{2-\lambda_2}(\alpha^{(2)} - \varepsilon^2\lambda_2) \end{aligned} \tag{3.18}$$

In (3.15)–(3.17) we take the limit (3.14) for the case when $\lambda \neq 2$, using the first equations of (3.18). Then

$$\frac{C_{11}^{(1)}}{\lambda_1 - \lambda_2} \to \pi_1^0, \quad C_{11}^{(2)} \to \frac{3L}{2(1+n^*)(2-\lambda_1^0)}, \quad C_{12}^{(1)} \to O(\varepsilon^2), \quad \frac{C_{22}^{(1)}}{\lambda_1 - \lambda_2} \to \pi_2^0$$

etc. Retaining terms with coefficients of the order of unity as $\varepsilon \to 0$, we obtain

$$\begin{aligned} p_{1\alpha\beta} &= \left[p_{1\alpha\beta}(0) - \frac{E}{\lambda_1^0 - 2}\right]\exp(-\lambda_1^0 t^*) + \frac{E}{\lambda_1^0 - 2}\exp(-2t^*) \\ p_{2\alpha\beta} &= p_{2\alpha\beta}(0)\exp(-\gamma_2\varepsilon t^*), \quad E = \frac{3LpU}{2(1+n^*)} \end{aligned} \tag{3.19}$$

The quantities λ_1^0, λ_2^0 are given by formulae (3.14).

We will now consider the case when $\lambda_1 = 2$, $\lambda_2 \neq 2$. We linearize

$$\exp(-\lambda_1 t^*) \approx \exp(-2t^*)[1 + (2-\lambda_1)t^*] \tag{3.20}$$

We substitute this expression into (3.15), taking into account the second equalities of (3.18), and then let $\lambda_1 \to 2$. We obtain

$$\begin{aligned} \pi_1 &= \frac{1}{2-\lambda_2}\{[C_{11}^{(1)} - G]e^{-2t^*} + [C_{12}^{(1)} + G]e^{-\lambda_2 t^*}\} - \alpha^{(3)}(\alpha^{(1)} - 2)Ut^*e^{-2t^*} \\ G &= \alpha^{(3)}(\alpha^{(1)} - \lambda_2)U \end{aligned} \tag{3.21}$$

and a similar expression for π_2.

In expression (3.21) all the coefficients are calculated for the same values of n^*, γ_i and ε for which $\lambda_1 = 2$.

In the limiting case (3.14) we obtain

$$p_{1\alpha\beta} \approx p_{1\alpha\beta}(0)\exp(-2t^*) + Et^*\exp(-2t^*) \tag{3.22}$$

We obtain the same result from the first equation of (3.19) using procedure (3.20). The expression for $p_{2\alpha\beta}$ is identical with the second expression of (3.19).

We will now analyse the results for the case when $\varepsilon \to 0$ and the remaining parameters of the problem are fixed. Suppose the quantities $h\Delta^2(0)$ and $E/p_{1\alpha\beta}(0)$ are negligibly small [13, 14]. We then obtain from formulae (3.7) and (3.19)

$$\begin{aligned} &\Delta_T \approx \Delta_T(0)\exp\left(-\frac{2\varepsilon^2 t^*}{x_2}\right) \\ &p_{1\alpha\beta} \approx p_{1\alpha\beta}(0)\exp(-\lambda_1^0 t^*), \quad p_{2\alpha\beta} \approx p_{2\alpha\beta}(0)\exp(-\varepsilon\gamma_2 t^*) \end{aligned} \tag{3.23}$$

We have the well-known pattern of the qualitatively different stages of relaxation [10, 12–15]. The stresses of the light component $p_{1\alpha\beta}$ relax together with the difference in the velocities Δ at the stage $t^* \sim 1$, and a stage of relaxation of the stresses of the heavy component $p_{2\alpha\beta}$ ($t^* \sim 1/\varepsilon$) then occurs, after which the temperatures of the components are equalized ($t^* \sim 1/\varepsilon^2$). At the next stage $\pi_i \ll 1$, i.e. the stresses are negligibly small. The heat fluxes behave similarly. We can therefore assume that the distribution functions at the third stage are close to Maxwellian with different T_i and the model of two-temperature gas dynamics [16] holds.

Relaxation of the stresses $p_{1\alpha\beta}$ is due to collision of the light particles with light and heavy particles, and the relaxation $p_{2\alpha\beta}$ is due solely to collision of the heavy particles. In fact, λ_1^0/τ_u depends on $n^* = n_2/n_1$, κ_{11}, κ_{12}, whereas λ_2^0/τ_u depends only on κ_{12}.

The situation changes if $h\Delta^2(0) \approx \Delta_T(0)/T$, $E \approx p_{1\alpha\beta}(0)$. Then formulae (3.23) are changed qualitatively: instead of $\Delta_T(0)$ we will have $\Delta_T(0) - h\Delta^2(0)T$, and relations (3.19) will hold. The system "remembers" the first stage. In the "resonance" case (3.22), relaxation is not purely exponential, and the solution includes the term $Et^*\exp(-2t^*)$.

It is important to bear in mind that what was said above about the distribution functions refers to their "domes", i.e. to the values of f_i for the thermal velocities of the molecules. It is these "domes" that determine the values of the lowest moments $M^{(n)}$ of the distribution functions, namely these moments occur in the equations of gas dynamics! The "tails" of the distribution functions relax more slowly the greater the value of ξ_i^2 (see Section 4).

In the equations for higher moments the number of non-uniform terms increase and, consequently, the possibility of non-exponential relaxation increases. For example, products of the vector Δ by π_i, π_j, Δ^2, Δ_T and T_i occur in the equation for $\mathbf{q}_i$. A similar analysis for $\varepsilon \ll 1$ shows that the new effect here is due to the term $\mathbf{\Delta}\Delta^2$. Instead of the condition $\lambda_1^0 = 2$ (see relation (3.14)) we have the condition $1 + (2/3)\gamma_1/n^* = 3$.

Above, as usual, when making the transition $\varepsilon \to 0$ we assumed the values of γ_i to be fixed. We will express κ_{ii} in terms of the coefficient of viscosity of the ith gas [2]

$$\mu_i = \left(\frac{2m_i}{\kappa_{ii}}\right)^{1/2}\frac{kT}{3\pi A_2} \tag{3.24}$$

and we will use the simplest combination rule

$$\kappa_{12} = \sqrt{\kappa_{11}\kappa_{22}} \tag{3.25}$$

Then

$$\gamma_1 = \left(\frac{\mu_2}{2\mu_1}\right)^{1/2}\varepsilon^{1/2}, \quad \gamma_2 = \left(\frac{\mu_1}{2\mu_2}\right)^{1/2}\varepsilon^{-1/2} \tag{3.26}$$

For inert gases μ_2/μ_1 depends slightly on the ratio of the masses, and the quantities γ_i depend on $\varepsilon^{1/2}$ [17]. We assumed above the $\varepsilon^{1/2}$ was of the order of unity. If $\varepsilon^{1/2} \ll 1$, then $p_{2\alpha\beta}$ decrease more rapidly: the stage of relaxation of the stresses of the heavy component will be $t^* \sim \varepsilon^{-1/2}$ instead of $t^* \sim \varepsilon^{-1}$. Then $\lambda_1^0 = 2B \neq 2$.

Using the above relations one can also consider other asymptotic forms with respect to the parameters of the problem (for example, $\varepsilon \to 0$ for fixed ρ_1/ρ_2).

4. SPATIALLY UNIFORM RELAXATION OF A BINARY MIXTURE (THE DISTRIBUTION FUNCTIONS)

The results presented in Section 3 for the lowest moments were obtained for arbitrary initial conditions and parameters of the mixture (with the requirement that f_i must be positive). The explicit solutions for the distribution function were obtained. naturally, with considerable limitations [18–21]. nevertheless, they demonstrate the most important feature of the process of translational relaxation: a considerable slowing down of the relaxation process as w_i^2 increases (i.e. on the “tails” of f_i). The definitions of the relaxation time usually employed relate to the “dome” of f_i (for $w_i^2 \approx 1$) [15].

The solutions for a simple gas [18, 19] were generalized to binary gas mixtures in [20] and multicomponent gas mixtures in [21]. For simplicity, the scattering of Maxwell molecules when they undergo collisions was assumed to be isotropic (in this case $A_2/A_2 = 2/3$ instead of 1.03). The conclusion was drawn in [21] that two relations between the parameters of the mixture instead of one were obtained erroneously [20]. The purpose of this section is to find an accurate solution for a binary mixture of monatomic gases without assuming them to be isotropic, to check the conclusion reached in [21], and to give a more complete analysis of the solution.

We will seek a solution in the form [20, 21]

$$\begin{gathered} f_i = n_i\left(\frac{m_i}{2\pi kT\theta(t)}\right)^{3/2}\exp(-w_i^2)[1+\alpha_i(t)S_{1/2}^{(1)}(w_i^2)] \\ \alpha_i = \frac{\theta-1}{\theta}\beta_i, \quad w_i^2 = \frac{m_i\xi_i^2}{2kT\theta(t)}, \quad S_{1/2}^{(1)}(w_i^2) = \frac{3}{2}-w_i^2 \end{gathered} \tag{4.1}$$

Here β_i is independent to t. The densities n_i and the temperature of the mixture T are also constant (by virtue of the conservation equations).

It is necessary to obtain θ and β_i and the relation between the parameters, which we will call the compatibility conditions.

By definition, the temperatures of the components of the mixture

$$T_i = \left(\frac{3}{2}n_ik\right)^{-1}\int\frac{m_i}{2}\xi_i^2 f_i d\xi_i = T\theta(1-\alpha_i), \quad T = \sum x_iT_i = \text{const}$$

(the latter follows from relation (1.12) with $\mathbf{u}_1 = \mathbf{u}_2$), and hence

$$x_1\beta_1 + x_2\beta_2 = 1, \quad T_i/T = \theta-(\theta-1)\beta_i \tag{4.2}$$

It follows from the last formula that the relaxation will be two-temperature relaxation ($T_1 \neq T_2$), if $\beta_1 \neq \beta_2$.

We will substitute expression (4.1) into the Boltzmann–Maxwell system of equations. After standard reduction, we obtain for the case of Maxwell molecules [2]

$$\begin{aligned} S_{1/2}^{(1)}\frac{\beta_i-1}{\theta}\frac{d\theta}{dt} - 2S_{1/2}^{(2)}\frac{\alpha_i}{\theta}\frac{d\theta}{dt} &= 4\pi S_{1/2}^{(1)}\phi_{12}^{(1)}nM_1M_2(1-\beta_i)\frac{\theta-1}{\theta} + \\ &+ 2S_{1/2}^{(2)}\alpha_i n\pi\left\{\phi_{ii}^{(2)}\frac{x_i\alpha_i}{2} + x_j\alpha_j 4M_1M_2[\phi_{12}^{(2)}2M_1M_2 + \phi_{12}^{(1)}(M_1-M_2)^2]\right\} \end{aligned} \tag{4.3}$$

$$\phi_{ii}^{(2)} = \left(\frac{2\kappa_{ii}}{m_i}\right)^{1/2} A_2 = \frac{2kT}{3\pi\mu_i}, \quad [\phi_{12}^{(1)}, \phi_{12}^{(2)}] = \left(\frac{m_0\kappa_{12}}{m_1 m_2}\right)^{1/2} [A_1, A_2]$$
$$i = 1, 2, \quad j \neq i = 1, 2, \quad M_i = m_i/m_0, \quad m_0 = m_1 + m_2 \tag{4.4}$$

The right-hand side of Eq. (4.3) is due to the collision integrals. In its first term we have used the equality

$$x_j(\beta_j - \beta_i) = 1 - \beta_i$$

which follows from the first formula of (4.2) and $x_1 + x_2 = 1$. In the first formula of (4.4) we have used formula (3.24), and $S_{1/2}^{(n)} = S_{1/2}^{(n)}(w_i^2)$ are Sonin polynomials, $n = 1.2$. We will introduce

$$t^* = \frac{\pi n}{2}\phi_{22}^{(2)} t = \frac{1}{3}\frac{p}{\mu_2} t \tag{4.5}$$

(the ratio μ_2/p is proportional to the relaxation time of a simple heavy gas, equal to μ_2/p_2). We will put

$$\Lambda = B(1 + \delta), \quad B = 8M_1M_2\frac{A_1}{A_2}G, \quad \delta = 2M_1M_2\left(\frac{A_2}{A_1} - 2\right)$$
$$G = \left(\frac{\kappa_{12}}{2M_1\kappa_{22}}\right)^{1/2} = \left(\frac{\mu^*}{2}\right)^{1/2}(M_1M_2)^{-1/4}, \quad \mu^* = \frac{\mu_2}{\mu_1} \tag{4.6}$$

In the expression for G we have used the simplest combination rule (3.24), and formula (3.24).

We equate the expression with $S_{1/2}^{(n)}$ in (4.3) to zero, assuming θ and β_i to be finite, with $\theta \neq 1, x_i \neq 0$, $M_i \neq 0$. Taking relations (4.4)–(4.6) into account, we obtain

$$\frac{1}{1-\theta}\frac{d\theta}{dt^*} = x_i\beta_i\frac{\mu_2}{\mu_i} + x_j\beta_j\Lambda \tag{4.7}$$

Subtracting and adding Eqs (4.7) with $i = 1$ and $i = 2$, we obtain

$$x_1\beta_1(\mu^* - \Lambda) = x_2\beta_2(1 - \Lambda) \tag{4.8}$$

$$\frac{1}{1-\theta}\frac{d\theta}{dt^*} = \lambda^*, \quad \lambda^* = \frac{1}{2}[x_1\beta_1(\mu^* + \Lambda) + x_2\beta_2(1 + \Lambda)] \tag{4.9}$$

whence

$$\theta = 1 + (\theta_0 - 1)\exp(-\lambda^* t^*), \quad \theta_0 = \theta(t = 0) > 0 \tag{4.10}$$

It follows from the first formula of (4.2) that $\beta_1 = 1$, if $\beta_2 = 1$, and vice versa. In the case when $\beta_i = 1$ the coefficients of $S_{1/2}^{(n)}$ in (4.3) are equal to zero, and from (4.8) we obtain a compatibility condition similar to that obtained previously in [20, 21] and which allows of the case $M_1 \ll 1$, when $x_1\mu^* \approx x_2$.

We will now consider the case when $\beta_i \neq 1$. After reduction by $1 - \beta_i$, the coefficients of $S_{1/2}^{(n)}$ in (4.3) are reduced to a single equation

$$\frac{1}{1-\theta}\frac{d\theta}{dt^*} = B \tag{4.11}$$

By virtue of equalities (4.9) and (4.11)

$$\lambda^* = 2B \tag{4.12}$$

To determine $x_1\beta_1$ and $x_2\beta_2$ we have the three equations (4.2), (4.8) and (4.12) (taking into account the definition of λ^* (4.9)). Eliminating, for example, $x_2\beta_2$ using the first formula of (4.2), we obtain from Eqs (4.8) and (4.12) respectively

$$x_1\beta_1 = \frac{\Lambda - 1}{\chi} = \frac{2B - \Lambda - 1}{\mu^* - 1}, \quad \chi = 2\Lambda - 1 - \mu^* \tag{4.13}$$

From the last equation we obtain $\Lambda^2 - \mu^* = B\chi$, whence, using the notation (4.6), we have the quadratic equation for B

$$(1-\delta^2)B^2 - (1+\mu^*)B + \mu^* = 0 \tag{4.14}$$

Hence, we have the following conditions of compatibility when $\beta_i \neq 1$:

$$B = \begin{Bmatrix} B_+ \\ B_- \end{Bmatrix} \equiv \frac{1}{\sigma}\left[1 \pm \left(1 - \frac{\mu^*\sigma^2}{1-\delta^2}\right)^{1/2}\right], \quad \sigma = 2\frac{1-\delta^2}{1+\mu^*} \tag{4.15}$$

$$\chi \neq 1, \quad \mu^* \neq 1 \tag{4.16}$$

They relate the parameters κ_{ii}, κ_{12}, M_i and do not depend on x_i. From the fact that f_i is positive we also have the requirements

$$1 + \frac{3}{2}\left(1 - \frac{1}{\theta_0}\right)\beta_i > 0 \quad \left(1 - \frac{1}{\theta_0}\right)\beta_i < 0 \tag{4.17}$$

The quantity θ_0 is defined by the second relation of (4.10).

It follows from inequalities (4.17) that β_i must have the same signs.

We will analyse condition (4.15) taking this requirement into account. It is clearly a contradiction when $M_1 \ll 1$: for example, for commensurable κ_{12} and κ_{22} and $\mu^* > 1$ we have from the second expression of (4.6) $B \ll 1$, and the right-hand side of the expression for B (4.15) is greater than or of the order of unity. We will initially assume that the combination rule (3.25) holds. Substituting the corresponding formula of (4.6) for B into condition (4.15), we obtain equations relating M_1 and μ^*,

$$8[M_1(1-M_1)]^{3/4}\frac{A_1}{A_2}\left(\frac{\mu^*}{2}\right)^{1/2} = \frac{1}{\sigma}\left[1 \pm \left(1 - \frac{\mu^*\sigma^2}{1-\delta^2}\right)^{1/2}\right]$$

Their numerical solutions show that β_i have the same signs for $B = B_-$. The same conclusion is reached if we do not use rule (3.25): by specifying the values of μ^* and M_1 in the sections [0.25, 4] and [0.1, 0.5] respectively, we find κ_{12}/κ_{22}, etc. For example, for an He–Ne mixture at $T = 300$ K we have $M_1 = 0.165$, $\mu^* = 1.593$, $B_+ = 1.879$ and $B_- = 0.913$. When $B = B_+$ the coefficients β_i have different signs. When $B = B_-$ we obtain $(\kappa_{12}/\kappa_{22})^{1/2} = 0.491$, $x_1\beta_1 = 0.264$, $\chi = -1.254$, $0.688 < \theta_0 < 1$ (the last for $x_i = 1/2$); these data are used in Fig. 1 for $x_i = 1/2$ and $\theta_0 = 0.8$.

We will put (see (4.1) and (4.5))

$$g_i = \frac{f_i}{f_i(t=\infty)} = \theta^{-3/2}\left[1 - \beta_i\left(\frac{1}{\theta} - 1\right)\left(\frac{3}{2} - \frac{\upsilon^2}{\theta}\right)\right]\exp\left[\upsilon^2\left(1 - \frac{1}{\theta}\right)\right], \quad \upsilon^2 = \frac{m_i\xi_i^2}{2kT}$$

where $f_i(t = \infty)$ is the Maxwell function. As $t \to \infty$ the functions g_i tend to unity more slowly the greater the value of υ^2.

Figure 1 illustrate this. In the figure the continuous curves give the values of g_1 when $\beta_1 = 0.528$, and the dashed curves give g_2 when $\beta_2 = 1.472$ for $t^* = 1/2$, 2 and 4 (the value of t^* is found from formula (4.5)). When $t^* = 8$ and $\upsilon < 10$, the values of g_i are close to unity.

When $\beta_i = 1$ the graphs of g_i against $\lambda^* t^*$ are the same as for a single-component gas [15, 19] (for the same value of θ_0).

5. HOMOENERGETIC AFFINE FLOWS

In the case of Maxwell molecules, the infinite system of equations of moments (1.3) can be split into recurrent closed systems of linear inhomogeneous equations with variable coefficients, if $M^{(n)}_{i\{\alpha\}}$ depend only on t, and consequently

$$\frac{D\mathbf{u}}{Dt} = \frac{1}{\rho}\sum \rho_i \mathbf{F}_i \tag{5.1}$$

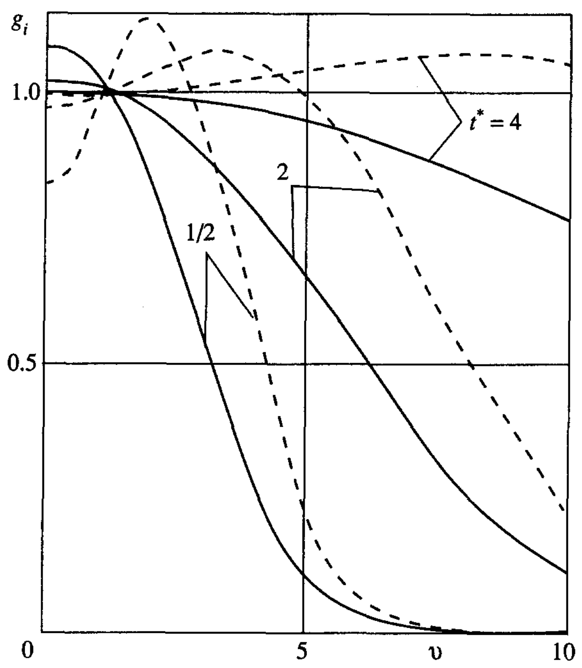

Fig. 1

The coefficients of Eq. (1.1) must be functions of t, so that for this class of flows

$$u_\alpha = \psi_{\alpha\beta}(t) r_\beta + \varphi_\alpha(t), \quad M^{(n)}_{i\{\alpha\}} = M^{(n)}_{i\{\alpha\}}(t), \quad n \ge 0 \tag{5.2}$$

(we recall that $M^{(n)}_{i\{\alpha\}} = 0$. Equation (1.1) reduces to the form

$$\begin{gathered}\frac{\partial f_i}{\partial t} - \frac{\partial f_i}{\partial C_{i\beta}} C_{i\gamma}\psi_{\beta\gamma} + \frac{\partial f_i}{\partial C_{i\beta}}\left[F_{i\beta} - \frac{1}{\rho}(\rho_1 F_{1\beta} + \rho_2 F_{2\beta} + \ldots + \rho_N F_{N\beta})\right] = J_i \\ f_i = f_i(\mathbf{C}_i, t), \quad f_i(\mathbf{C}_i, 0) = g_i(\mathbf{C}_i) > 0\end{gathered} \tag{5.3}$$

where g_i are specified functions of $\mathbf{C}_i$. If the partial external forces $\mathbf{F}_i$, referred to m_i, are the same and equal to $\mathbf{F}$, the coefficient of the third term on the left-hand side is equal to zero (the last term in (1.7) vanishes as a consequence of the equality $\sum \rho_i \mathbf{V}_i = 0$). The quantity $\mathbf{F}$ is represented by a linear function of the coordinates, and the necessary relations between the coefficients are found from Eq. (5.1), taking the first formula of (5.2) into account.

A more interesting case is when $\mathbf{F}_i$ are different. Then $\mathbf{F}_i = \mathbf{F}_i(t)$. The function $\varphi_\alpha(t)$ are found from the system of equations

$$\frac{d\varphi_\alpha}{dt} + \psi_{\alpha\beta}\varphi_\beta = \sum \frac{\rho_i(t)}{\rho(t)} F_{i\alpha}(t)$$

and the coefficients $\psi_{\alpha\beta}(t)$ are given by the formulae [22]

$$\begin{gathered}\psi_{\alpha\beta} = \{|a_{\gamma\delta}| t^2 \delta_{\alpha\beta} + (a_{\gamma\gamma} a_{\alpha\beta} - a_{\alpha\gamma} a_{\gamma\beta}) t + a_{\alpha\beta}\} H^{-1}, \quad a_{\alpha\beta} = \psi_{\alpha\beta}(0) \\ H \equiv |\delta_{\gamma\delta} + a_{\gamma\delta} t| = |a_{\gamma\delta}| t^3 + 1/2(a_{\gamma\gamma} a_{\delta\delta} - a_{\gamma\delta} a_{\delta\gamma}) t^2 + a_{\gamma\gamma} t + 1\end{gathered} \tag{5.4}$$

Here $|a_{\gamma\delta}|$ is the determinant of the elements $a_{\gamma\delta}$ while $\delta_{\alpha\beta}$ is the Kronecker delta, and we have used the rule of summation over repeated subscripts α, β, γ, $\delta = 1, 2, 3$. Formulae (5.4) are found from the system of equations $D(\psi_{\alpha\beta} r_\beta)/Dt = 0$.

From the equation of continuity we obtain

$$\rho = \rho(0)H^{-1} \tag{5.5}$$

If $\varphi = 0$ we have $\Sigma\rho_i\mathbf{F}_i = 0$. In this case, neglecting the collision operators J_i, we obtain [23] the "free-molecule" solution of problem (5.3)

$$\begin{aligned} f_i &= g_i(E_{ix}, E_{iy}, E_{iz}) \\ E_{i\alpha} &= (\delta_{\alpha\beta} + a_{\alpha\beta}t)C_{i\beta} - \int_0^t (\delta_{\alpha\beta} + a_{\alpha\beta}t)F_{i\beta}dt \end{aligned} \tag{5.6}$$

We will now review the moment solutions. Solutions which enable problems of the applicability of the Navier–Stokes, Burnett, etc. approximations to be considered are of paramount interest. The main part of previous paper is devoted to stresses in a simple gas. Solutions of the equations for the diffusion velocities, due to the difference in $\mathbf{F}_i$, were obtained in [22] in the case of a binary gas mixture.

Henceforth we will confine ourselves to a simple gas of Maxwell molecules when there are no external forces, assuming $\varphi = 0$.

We will write the energy equation in the form

$$\frac{dp}{dt} + \frac{5}{3}\psi_{\alpha\alpha}p + \frac{2}{3}\psi_{\alpha\beta}p_{\beta\alpha} = 0 \tag{5.7}$$

For the stresses $p_{\alpha\beta}$ and the third-order moments $Q_{\alpha\beta\gamma}$ we obtain

$$\frac{dp_{\alpha\beta}}{dt} + p_{\alpha\beta}\psi_{\gamma\gamma} + 2\langle p_{\alpha\gamma}\psi_{\beta\gamma}\rangle + 2p\langle\psi_{\alpha\beta}\rangle + \frac{1}{H\tau}p_{\alpha\beta} = 0 \tag{5.8}$$

$$\begin{aligned} &\frac{dQ_{\alpha\beta\gamma}}{dt} + \psi_{\gamma\delta}Q_{\alpha\beta\delta} + \psi_{\beta\delta}Q_{\alpha\delta\gamma} + \psi_{\alpha\delta}Q_{\delta\beta\gamma} + \psi_{\delta\delta}Q_{\alpha\beta\gamma} + \\ &+ \frac{1}{3H\tau}\left(\frac{9}{2}Q_{\alpha\beta\gamma} - q_\alpha\delta_{\beta\gamma} - q_\beta\delta_{\alpha\gamma} - q_\gamma\delta_{\alpha\beta}\right) = 0 \end{aligned} \tag{5.9}$$

$$\tau = \frac{\mu_0}{R\rho(0)} = \frac{\mu(0)}{p(0)}, \quad R = \frac{k}{m}$$

We have used relations (1.8)–(1.10) here; R is the gas constant.

The initial values $p(0)$, $p_{\alpha\beta}(0)$, $Q_{\alpha\beta\gamma}(0)$ when $t = 0$ are specified. The quantity τ is the translational relaxation time, calculated from the initial values.

System (5.7), (5.8) closes the problem of determining the gas-dynamic variables, i.e. p for specified ρ and $\mathbf{u}$. It is useful to determine

$$\Pi_{\alpha\beta} = p_{\alpha\beta}/p, \quad \Pi_{\alpha\alpha} = 0 \tag{5.10}$$

It follows from Eq. (5.7) that

$$p = p(0)H^{-5/3}\Pi, \quad \Pi = \exp\left(-\frac{2}{3}\int_0^t \psi_{\alpha\beta}\Pi_{\beta\alpha}dt\right) \tag{5.11}$$

Assuming $\Pi = 1$, we obtain p in the Euler approximation. For $\Pi_{\alpha\beta}$ we obtain a closed but more complex system of equations

$$\frac{d\Pi_{\alpha\beta}}{dt} - \frac{2}{3}\Pi_{\alpha\beta}(\psi_{\gamma\delta}\Pi_{\delta\gamma} + \psi_{\gamma\gamma}) + 2\langle\Pi_{\alpha\gamma}\psi_{\gamma\beta}\rangle + 2\langle\psi_{\alpha\beta}\rangle + \frac{\Pi_{\alpha\beta}}{H\tau} = 0 \tag{5.12}$$

Investigations of shear flow were the origin of this problem [24, 25]. After this, solutions with an unsteady velocity were investigated [26–32, 23, 23]. Many results are summarized and supplemented in [1] (see also [32]). Theorems of existence and uniqueness of the solution of this initial problem for the Boltzmann–Maxwell equation were proved in [4] (taking into account the condition $\mathbf{F}$ = const).

Equations (5.7)–(5.12) have, generally speaking, been solved numerically. In the papers mentioned, particular solutions of systems of these equations were found in terms of elementary functions, when they were reduced to a system of linear equations with constant coefficients. The so-called dominant [24] exact solutions, i.e. solutions for those initial conditions when the initial boundary (kinetic) layer disappears, were of particular interest. For such solutions comparisons of the individual approximations of the Chapman–Enskog and Hilbert methods with one another and with the exact solution are obvious.

The initial boundary layer is due to the fact that Eqs (5.12) for a Knudsen number $\mathrm{Kn} = \tau/t_0 \ll 1$ are equations with a small parameter with a derivative, where t_0 is the characteristic time. For these special solutions the Chapman–Enskog series is a power series with constant coefficients, and hence the dominant solutions are determined by the requirement $\Pi_{\alpha\beta}$ = const. From system (5.12) we obtain algebraic equations defining $\Pi_{\alpha\beta}$ for and t and Kn. It was proved in [1] that, by expanding in $\mathrm{Kn} \ll 1$, we obtain a Chapman–Enskog series for $\Pi_{\alpha\beta}$, which converges for sufficiently small Kn. Substituting this series into expression (5.11), we obtain a solution of the gas-dynamic problem using the equations of the Chapman–Enskog method. By expanding the function Π in terms of Kn for fixed t, we obtain a solution by the Hilbert method.

It is interesting to obtain and investigate the dominant solutions for the special case when the Chapman–Enskog series for $\Pi_{\alpha\beta}$ depends on t and is asymptotic. This occurs in the case of plane outflow–inflow [22], when only $a_{11} = a_{22}$ are non-zero, i.e.

$$u_1 = \frac{x_1}{t+c}, \quad u_2 = \frac{x_2}{t+c}, \quad u_3 = 0, \quad H = (1+t^*)^2, \quad \rho = \frac{\rho(0)}{(1+t^*)^2}$$
$$c = \frac{1}{a_{11}}, \quad t^* = \frac{t}{c} = a_{11}t, \quad t \in [0, t_f], \quad t_f = \begin{cases} \infty, & a_{11} > 0 \\ 1/|a_{11}|, & a_{11} < 0 \end{cases} \tag{5.13}$$

From relations (5.11) we have

$$p = p(0)(1+t^*)^{-10/3}\Pi, \quad \Pi = \exp\left(\frac{2}{3}\int_0^{t^*} \frac{\Pi_{33}}{1+t^*}dt^*\right) \tag{5.14}$$

To solve the gas-dynamic problem it is sufficient to determine Π_{33}, for which it follows from Eqs (5.12) that

$$\frac{d\Pi_{33}}{dt} + \frac{2}{3(1+t^*)}(\Pi_{33}^2 - \Pi_{33} - 2) + \frac{\Pi_{33}}{\beta(1+t^*)^2} = 0$$
$$\beta = \mathrm{Kn}\,\mathrm{sign}\,a_{11}, \quad \mathrm{Kn} = |a_{11}|\tau, \quad \tau = \mu_0/(R\rho(0)) \tag{5.15}$$

The quantity c is chosen as the characteristic time. We recall that in this case the viscosity coefficient $\mu = \mu_0 T$, μ_0 = const.

We will obtain the hydrodynamic limit as $\mathrm{Kn} \to 0$ and t^*, $p(0)$ and $\Pi_{\alpha\beta}(0)$ are fixed. For fixed t and a_{11} this limit is reached by reducing τ, i.e. the kinematic viscosity $\mu_0/\rho(0)$.

Introducing the variables

$$\zeta = (1+t^*)\beta, \quad u(\zeta) = \exp\left(\frac{2}{3}\int \Pi_{33}\frac{d\zeta}{\zeta}\right)$$

and then

$$z = 1/\zeta, \quad u = z^{2/3}y(z)$$

we reduce Eq. (5.15) to the standard form

$$z\frac{d^2y}{dz^2} + (3-z)\frac{dy}{dz} - \frac{2}{3}y = 0 \tag{5.16}$$

The general solution of Eq. (5.16) is

$$y = C_1\Phi(2/3, 3; z) + C_2\Psi(2/3, 3; z)$$

where Φ and Ψ are degenerate hypergeometric functions of the second integer argument [33] (in the notation employed previously in [34] these are the functions M and U respectively). Consequently

$$\Pi_{33} = -1 - \frac{3}{2}z\frac{C\Phi' + \Psi'}{C\Phi + \Psi}, \quad \Phi' = \frac{d\Phi(2/3, 3; z)}{dz} \tag{5.17}$$

and the arbitrary constant C is found from the condition $\Pi_{33}(z = 1/\beta) = \Pi_{33}(0)$.

We will first obtain the dominant solution for outflow ($a_{11} > 0$). When $\beta \to 0$ the function Π_{33} must tend to the Navier–Stokes value $4/3z$). For this solution we have $z \to \infty$ as $\beta \to 0$. In the neighbourhood of an infinitely distant point Φ and Ψ can be expanded in asymptotic series [33]

$$\Phi = e^z z^{-7/3}[1 + O(1/z)], \quad \Psi = z^{-2/3}[1 + O(1/z)]$$

Further, in fact following the principle of minimum singularity, we assume $C = 0$, in which case

$$\Pi^d_{33} = -1 - \frac{3}{2}z\frac{d}{dz}\ln\Psi\left(\frac{2}{3}, 3; z\right), \quad z^{-1} = \mathrm{Kn}(1 + t^*) \tag{5.18}$$

We will denote the dominant solution by the superscript d. The expansion of the quantity (5.18) with respect to large z (small β) is

$$\Pi^d_{33} = \frac{4}{3}z^{-1} - \frac{4}{9}z^{-2} - \frac{16}{27}z^{-3} + \frac{176}{81}z^{-4} + \ldots, \quad z^{-1} = \beta(1 + t^*) \tag{5.19}$$

Series (5.19) is asymptotic, since the ratio of the $(n + 1)$th term to the nth term is of the order of nz^{-1}.

A similar analysis for the case of inflow ($\beta < 0$) shows that here

$$\Pi^d_{33} = -1 - \frac{3}{2}z\frac{d}{dz}\ln\Phi\left(\frac{2}{3}, 3; z\right), \quad z^{-1} = -\mathrm{Kn}(1 + t^*) \tag{5.20}$$

The expansion of (5.2) with respect to large z is given, naturally, by formula (5.19) with $\beta = -\mathrm{Kn}$.

The right-hand part of Fig. 2 corresponds to outflow ($\beta > 0$) and the left-hand part corresponds to inflow ($\beta < 0$). The exact values of Π^d_{33} (5.18), (5.20) are given by the continuous curve. The approximate values of Π^d_{33}, obtained using series (5.19), are represented by the dashed curves. The first term of expansion (5.19) is given by the Navier–Stokes approximation (line 1 in Fig. 2), the first two terms are

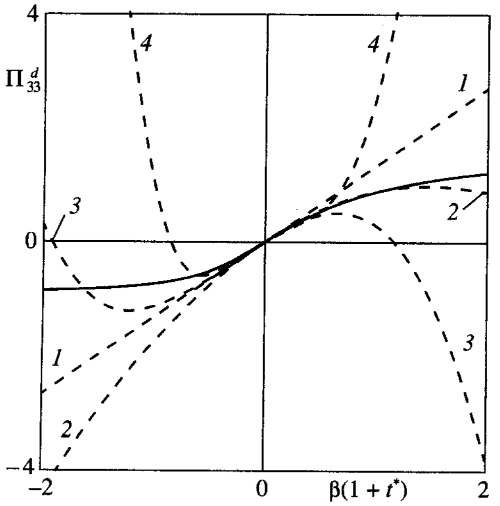

Fig. 2

given by the Burnett approximation (2), the first three terms are given by the super-Burnett approximation (3), and curve 4 corresponds to the terms taken into account in (5.19).

In the case of outflow, the Burnett approximation gives the best agreement with the exact solution, but for inflow the Burnett approximation gives a worse agreement with the exact solution than the Navier–Stokes approximation.

A similar example of the high accuracy of the Burnett approximation was indicated for the first time in [27] (the exact solution for one-dimensional outflow), and for the other solutions in [31, 32]. The evolution of opinions on the significance of Burnett's equations and examples of Burnett effects are described in the review [35] (see also [36]).

We can now obtain the pressure, i.e. the function Π. Substituting expansion (5.19) into expression (5.14), we obtain the Chapman–Enskog solution

$$\begin{aligned}\Pi = \exp\Big\{&\frac{8}{9}\beta t^* - \frac{4}{27}\beta^2 t^*(2+t^*) - \frac{32}{81}\beta^3 t^*\Big(1+t^*+\frac{1}{3}t^{*2}\Big) + \\ &+ \frac{176}{243}\beta^4 t^*(2+t^*)\Big(1+t^*+\frac{1}{2}t^{*2}\Big) + \ldots\Big\}\end{aligned} \tag{5.21}$$

It is important to emphasise that the Chapman–Enskog series for $p_{33} = p\Pi_{33}$ is not a power series in Kn. This is a general property of the Chapman–Enskog method, unlike the Hilbert method [4]. We obtain a solution by the Hilbert method by expanding the right-hand side of relation (5.21) in series

$$\begin{aligned}\Pi = {}& 1 + \frac{8}{9}\beta t^* + \frac{4}{81}\beta^2 t^*(5t^* - 6) - \\ &- \frac{32}{2187}\beta^3 t^*[27 + 5t^*(9+2t^*)] + \frac{16}{19683}\beta^4 t^*[1782 + \ldots]\end{aligned} \tag{5.22}$$

Substituting expression (5.18) or (5.20) into relation (5.14), we obtain the dominant solution for the pressure p. For example, for outflow

$$\Pi^d = (1+t^*)^{-2/3} z\Phi\Big(\frac{2}{3}, 3; -\frac{1}{|\beta|(1+t^*)}\Big)\Phi^{-1}\Big(\frac{2}{3}, 3; -\frac{1}{|\beta|}\Big) \tag{5.23}$$

Expanding expression (5.23) in terms of Kn ≪ 1, we arrive at relation (5.22).

We will relate the values of Π, given by the Navier–Stokes equations and corresponding to the approximation of the Hilbert method, to the value of Π^d. In the case of outflow, we obtain respectively

$$\sigma_1 = \frac{1}{\Pi^d}\exp\Big(-\frac{8}{9}\mathrm{Kn}t^*\Big), \quad \sigma_2 = \frac{1}{\Pi^d}\Big(1 - \frac{8}{9}\mathrm{Kn}t^*\Big), \quad t^* = \frac{t}{c}$$

where the function Π^d is given by formula (5.23). The values of the ratios σ_1 and σ_2 are represented by the dashed and continuous curves in Fig. 3, respectively.

The Navier–Stokes equations have much higher accuracy, since σ_1 is much closer to unity than σ_2.

Using the example of shear flow it was shown in [32] that the solutions in the first Chapman–Enskog (Navier–Stokes) approximation and the third Chapman–Enskog approximation agree much better with the exact solutions than the solutions in the corresponding Hilbert approximations. Certain researchers [4] proposed to modify the Hilbert method in such a way that, in the zeroth approximation, the Navier–Stokes (rather than the Euler) equations are obtained, and further, a sequence of inhomogeneous linearized Navier–Stokes equations. This modification in fact leads to a refinement of the Hilbert method [32].

In general, the constant C is non-zero and is determined by the initial value of $\Pi_{33}(0)$, on which, generally speaking, the comparative accuracy of the different approximate methods depends. The different of C from zero for small Kn is a measure of the effect of the initial kinetic (Knudsen) layer. The expansion of C in series as $\beta \to 0$ is obtained by the method of matched asymptotic expansions, while the Hilbert series is an outer expansion. A comparison of the results obtained by the Chapman–Enskog and Hilbert methods, using the example of a non-dominant exact solution, has been given in [30] for one-dimensional outflow.

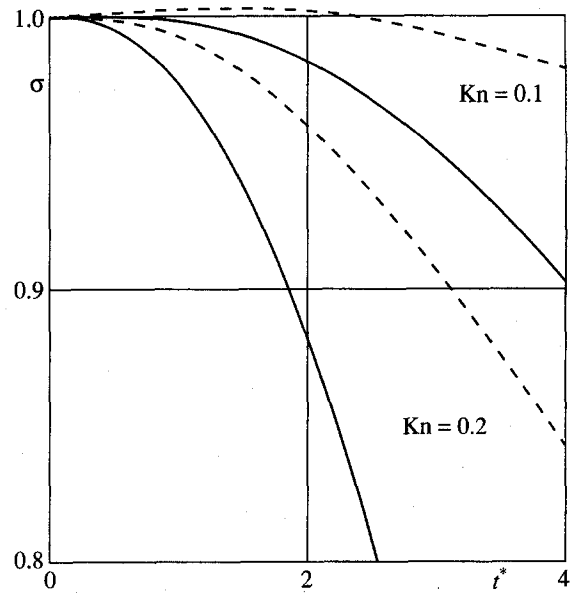

Fig. 3

Considerable attention has been devoted [1, 24] to constructing iterative schemes, which differ from that which the Chapman–Enskog expansion gives. The regrouping of the Chapman–Enskog series obtained in this way may turn out to be effective: several terms of the "regrouped" series give a better approximation to the exact solution for shear flow than the corresponding sections of the Chapman–Enskog series [1]. However, this result is only a particular one [30].

The main attention, in the publications mentioned, has been devoted to the gas-dynamic limit $\beta \to 0$, but other asymptotic forms were also considered [1, 22–24]: expansions of the solutions about the free-molecular solution, in small $a_{\alpha\beta}$ (the effect of small velocity gradients on the translational relaxation process), etc. Interesting results were obtained in [28, 29, 31] when analysing the asymptotic behaviour of certain exact and approximate solutions as $t \to \infty$. The exact solutions considered in these publications are asymptotically stable ($p \to 0$ as $t \to \infty$), but their Navier–Stokes approximations are asymptotically unstable ($p \to \infty$). The Burnett approximations improve the Navier–Stokes approximations, giving a qualitatively correct result ($p \to 0$) [31]. This is one example of the effectiveness of the Burnett equations.

A considerable number of papers on shear flow

$$u_1 = a_{12}x_2, \quad u_2 = 0, \quad u_3 = 0, \quad \rho = \text{const}, \quad a_{12} > 0$$

have been published [37–39]. However, additional assumptions were made together or separately: approximate models of the collision operator were used, and it was assumed that the external force is proportional to the peculiar velocity of the molecules (this force was introduced in order that the temperature of the shear flow should be finite as $t \to \infty$). As has already been pointed out in the introductory part of this paper, we do not consider this aspect of the subject due to limitations of space.

Considerable interest has been aroused [1, 37–39] by the peculiar "instability" of the third-order moments in shear flow [24]: for $\text{Kn} = a_{12}\tau > 3/\sqrt{2}$ and $t/\tau \to \infty$ the heat flux q_3 approaches infinity, and when $\text{Kn} < 3/\sqrt{2}$ it approaches zero. However, similar peculiarities in the behaviour of non-hydrodynamic moments are also shown by other flows of this class.

Consider one-dimensional outflow–inflow

$$u_1 = \frac{x_1}{t+c}, \quad u_2 = 0, \quad u_3 = 0, \quad \rho = \frac{\rho(0)}{1+t^*} \tag{5.24}$$

The quantities c, t^* and β are found from relations (5.13) and (5.15).

We obtain, for example, [32]

$$p_{13} = p_{13}(0)(1+t^*)^{-2-1/\beta}, \quad p_{23} = p_{23}(0)(1+t^*)^{-1-1/\beta} \tag{5.25}$$

For outflow ($\beta > 0$) p_{13} and p_{23} decrease as t^* increases, but for inflow ($\beta < 0$) this can only occur for $\beta < -1/2$ and $\beta < -1$ respectively.

Consider the third-order moments. For the flow (5.24) we introduce the variable $\eta = \ln(1 + t^*)$. We have from Eqs (5.9)

$$Q_{123} = Q_{123}(0)\exp\left[-\left(2 + \frac{3}{2\beta}\right)\eta\right]$$

For $(Q_{k11}, Q_{k22}, Q_{k33})$ with $k = 1, 2$ we obtain two homogeneous systems of ordinary differential equations with constant coefficients. Writing the solution in terms of $\exp(-\lambda\eta)$, for the eigenvalues λ we obtain the formulae

$$3 - k + \frac{3}{2\beta},\ \frac{1}{12\beta}\{13 + 12(4 - k)\beta \mp [25 + 24(4k - 5)\beta + 144\beta^2]^{1/2}\}$$

where $k = 1, 2$. The expressions for $k = 2$ also hold for the system $(Q_{311}, Q_{322}, Q_{333})$.

When $\beta > 0$ all λ are positive. When $\beta < 0$ the eigenvalues vanish for the following values of $|\beta|$ respectively: (0.75, 0.196, 0.637) for the eigenvalues with $k = 1$ and (1.5, 0.365, 0.912) for $k = 2$.

Hence, in the case of inflow third-order moments decrease for fairly high Kn, while for low Kn they increase (similar to the stresses (5.25)).

6. SPHERICAL OUTFLOW–INFLOW

The Nikol'skii transformation. The flow (2.5) is described by the locally Maxwell function (2.6), provided that the state is locally Maxwellian at the initial instant of time also.

This flow is unique in the class indicated in Section 2, which belongs to homoenergetic affine flows (uniform in the terminology used by Nikol'skii [40–42]). Unlike the other flows (5.2), the pressure satisfies Euler's equation

$$p = p(0)(1 + t^*)^{-5}$$

In the case of Maxwell molecules, we obtain [22] from system (5.8)

$$\Pi_{\alpha\beta} = \Pi_{\alpha\beta}(0)\exp\{[(1 + t^*)^{-2} - 1]/(2\beta)\}$$

Here we have used the notation (5.10) and (5.15).

We will fix $\Pi_{\alpha\beta}(0)$ and β. Then for outflow ($\beta > 0$) as $t^* \to \infty$ we have

$$\Pi_{\alpha\beta} \to \Pi_{\alpha\beta}(0)\exp\{-1/(2\beta)\} \neq 0$$

i.e. $\Pi_{\alpha\beta}$ approaches a finite value, and not the locally Maxwell value $\Pi_{a\beta} = 0$. In the case of inflow ($c < 0$, $\beta < 0$, $t^* = -t/|c|$) as $t^* \to -1$ we have $\Pi_{\alpha\beta} \to 0$.

In the Navier–Stokes, Burnett, etc. approximations, $\Pi_{\alpha\beta} = 0$. The latter makes spherical outflow–inflow of less interest (from the gas-dynamic point of view) compared with the remaining flows (5.2), for which certain $\Pi_{\alpha\beta} \neq 0$ in the Navier–Stokes approximation. However, in this case, it is possible to obtain interesting results for the distribution function.

Solution (2.6) suggests that one should introduce the variable $\mathbf{C}_i^* = (1 + t^*)\mathbf{C}_i$ instead of $\mathbf{C}_i$. In the variables t^*, $\mathbf{C}_i^*$ on the left-hand side of the kinetic equation, only the term with the partial derivative of $f_i(t^*, \mathbf{C}_i^*)$ with respect to t^* remains, and a factor appears in front of the collision operator which depends on ν and t^*. Eliminating this by introducing a new variable instead of t^*, we reduce Eq. (5.3) to the form [42]

$$\partial f_i^*/\partial Y = J_i^*, \quad f_i^* = f_i(Y, \mathbf{C}_i^*), \quad Y = Y(\nu, t^*) \tag{6.1}$$

(external forces are ignored).

It is assumed [42] that the intermolecular force is equal to $\kappa_{ij}r^{-\nu}$, $\kappa_{ij} = \text{const}$, r is the intermolecular distance, and the factor $\nu > 2$ is the same for all intermolecular interactions of particles of sorts i and j, where $i, j = 1, 2, \ldots, N$ (the case of molecules in the form of elastic spheres was considered previously in [41]). In the case of outflow when $t^* \to \infty$ and $\nu > 7/3$ the function Y approaches a finite limit Y_∞ and when $\nu \geq 7/3$ we have $Y \to \infty$ (for Maxwell molecules $\nu = 5$). In the case of inflow $Y_\infty = \infty$ when $\nu \geq 7/3$ and $Y_\infty \neq 0$ when $\nu < 7/3$.

As a result the following assertion holds [42]: suppose we are given the solution of problem (3.1) $f_i = f_i(t, \xi_i)$, which, as $t \to \infty$, approaches the Maxwell function $f_i(\infty, \xi_i) \to f_i^M(\xi_i^2)$. Then, for the same initial conditions, the solution of problem (6.1) $f_i^* = f_i(Y, \mathbf{C}_i^*)$ is known. When $t \to \infty$ the distribution function $f_i(Y, \mathbf{C}_i^*) \to f_i(Y_\infty, \mathbf{C}_i^*)$, i.e. it does not approach the locally Maxwell function f_i^M if $\nu > 7/3$ (outflow) and $\nu < 7/3$ (inflow), since $Y_\infty \neq \infty$, and $f_i \to f_i^M$ in the remaining cases when $Y_\infty = \infty$.

For flows of class (5.2) the conditions of applicability of the H-theorem [1, 6] are, generally speaking, not satisfied.

Using the results of Section 4 [18–21, 43, 44], we obtain explicit expressions for the distribution functions in the case of outflow–inflow.

A more general formulation of the problem is as follows [45]: when these are external forces in a simple gas one can formulate transformations which reduce the Boltzmann–Maxwell kinetic equation to the form (6.1). The flows obtained are the superposition of spherical outflow–inflow and rotation of the gas. When there are no external forces one obtains Nikol'skii's results [42].

7. THE DOMINANT (NORMAL) SOLUTIONS

We will continue the discussion (see Section 5) of the dominant exact solutions. For clarity we will consider the flows (5.24), when, according to relation (5.11), the pressure p is expressed in terms of the ratio Π_{11}, which satisfies the equation

$$3\beta\frac{d\Pi_{11}}{d\ln(1+t^*)} = 2\beta\Pi_{11}^2 - (2\beta+3)\Pi_{11} - 4\beta \tag{7.1}$$

where β is expressed in terms of the Knudsen number (5.15). Its solution is

$$\begin{gathered}\Pi_{11} = -\frac{5+r_1}{2}\left[1+\frac{5+r_2}{5+r_1}Q\right][1+Q]^{-1}\\ Q = \left[\frac{r_2-r_1}{5+r_2+2\Pi_{11}(0)}-1\right](1+t^*)^q, \quad q = \frac{r_2-r_1}{3}\\ r_{1,2} = -\frac{3}{2\beta}\left[1+4\beta\mp\sqrt{1+4\beta\left(\frac{1}{3}+\beta\right)}\right]\end{gathered} \tag{7.2}$$

For any $t^* \geq 0$ the following dominant solution holds

$$\Pi_{11}^d = \Pi_{11}(0) = -\frac{5+r_1}{2} \tag{7.3}$$

For outflow $\beta > 0$ and $q < 0$, and for inflow $\beta < 0$ and $q > 0$. Correspondingly, as $t^* \to \infty$ and $t^* \to -1$ we have $\Pi_{11} \to \Pi_{11}^d$, i.e. the solution approaches the dominant solution more rapidly the smaller the value of $|\beta|$. Expanding expression (7.3) in terms of small β, we obtain the Chapman–Enskog series (external expansion of the solution of Eq. (7.1)), in this case with constant coefficients, which converges absolutely for $|\beta| < 1/2$. (for a rigorous proof see, for example, [1].) Hence the dominant solution can also be obtained by assuming Π_{11} to be constant; from Eq. (7.1) we obtain a quadratic equation for Π_{11}, and the choice of the root is governed by the fact that the first term of the expansion must be the Navier–Stokes solution.

The properties of the solutions for certain other flows [24, 25, 31, 32] are similar. In the case of (5.13) the terms of the Chapman–Enskog series depend on t^*, and the series itself is asymptotic. According to the solution (5.18), Π_{33}^d depend only on $z = \zeta^{-1}$, and the quantity can be written in the form

$$\zeta = \frac{3}{4}\frac{p_{33}^{(1)}}{p} = \frac{\mu}{2p}\nabla\mathbf{u}$$

The stress $p_{33}^{(1)}$ is given by Navier–Stokes approximation.

Hence, the dominant solution (5.180 depends on t via the gas-dynamic variables, and hence (like the solution (7.3)) it can be called normal in the Hilbert sense. However, in kinetic theory it is the approximate solutions of the kinetic equations in the form of outer asymptotic expansions, each term of which is expressed in terms of the gas-dynamic variables and their spatial derivatives, that are usually called normal [1, 4, 6]. Algorithms of the expansions are known, unlike the algorithms for constructing dominant solutions.

The most important property of the dominant solution is the existence of non-linear local expressions for the transport properties. Their accuracy and area of application depend on the specific conditions. The detection of cases of degeneracy of the Chapman–Enskog series was the origin of investigations for real flows. The first case was apparently slow steady flows, described by the linearized Boltzmann–Maxwell kinetic equation (see the history of the problem in [46]). Non-linear examples for a gas of Maxwell molecules were given later: heat transfer between parallel plates [47] and plane Couette flow with heat transfer [48]. A review and an extension to a mixture of gases is given in [49].

It was shown in [47], by analysing an infinite chain of transport equations for the coefficients of the expansion of f in Hermie polynomials for $\mathrm{Kn} \ll 1$ that, in the case of heat transfer, the conservation equations in any order with respect to Kn reduce to equations in the Navier–Stokes–Fourier approximation (as also in [46])

$$p = \text{const}, \quad p_{\alpha\beta} = 0, \quad q_y = q_y^{(1)} = -\lambda_0 T\frac{dT}{dy} = \text{const}, \quad \lambda_0 = \text{const}$$

which give the dominant solution.

In the case of Couette flow [48] the structure of the expansion with respect to $\mathrm{Kn} \ll 1$ is established, in particular,

$$p_{xy}^* \equiv \frac{p_{xy}}{p_{xy}^{(1)}} = g_1(\varepsilon^2), \quad q_y^* \equiv \frac{q_y}{q_y^{(1)}} = g_2(\varepsilon^2), \quad \varepsilon = \frac{\mu}{p}\frac{du_x}{dy} = \text{const} \tag{7.4}$$

Here x and y are the coordinates along and transverse to the plates, the stress $p_{xy}^{(1)}$ and the heat flux $q_y^{(1)}$ are given by the Navier–Stokes–Fourier approximation, and g_1 and g_2 are power series in ε^2 with constant coefficients. The results for the problem of heat transfer [47] follow from these results when $u_x = 0$.

Within the framework of the model kinetic equation (the BGKV-model) it was possible to obtain dominant solutions (normal solutions in the terminology of a number of researchers [50–54]) of the problem of heat transfer [50] and on the problem of Couette flow [51] for the distribution function with diffuse reflection of molecules from the walls. To "liquidate" Knudsen layers, the temperatures of the plates were assumed to be equal to zero and infinity in [50] and zero in [51].

The results of calculations of this flow by the method of direct statistical modelling showed [52, 53], that outside the Knudsen layers over a wide range of values of the governing parameters, the quantities p_{xy}^* and q_y^* are functions of only $\varepsilon^2 \in [0. 2.8]$, with extremely high accuracy. The functions g_1 and g_2 were obtained; these varied from unity (the Navier–Stokes approximation) to ≈ 0.2.

The existence of non-linear local expressions for the transport properties (the considerable difference of g_1 and g_2 from unity) in the case of Couette flow can be explained qualitatively as follows. The Knudsen layers can be made fairly thin by reducing the temperatures of the plates. However, for a high Mach number of the moving plate, the temperature of the gas outside the layers will be high, and the parameter ε will be of the order of unity (p = const), so that the transport properties here will be local and differ considerable the Navier–Stokes approximations.

The result is one of the confirmations of the need to the search for models which differ from the Navier–Stokes model. During the course of this, on the basis of an analysis of the profiles of the macro-parameters in a storing shock wave in a monatomic case, calculated by the method of direct statistical modelling, it was possible to establish [54] approximate local relations for the transport properties

$$\frac{p_{xx}}{p} = F_p\left(\frac{\mu(T)}{p}\frac{du_x}{dx}, \frac{\lambda(T)}{pc}\frac{dT}{dx}\right), \quad \frac{q_x}{pc} = F_q\left(\frac{\mu(T)}{p}\frac{du_x}{dx}, \frac{\lambda(T)}{pc}\frac{dT}{dx}\right), \quad c = \sqrt{2\frac{k}{m}T}$$

which are extremely accurate. Here, the x axis is directed along the flow, μ and λ are the coefficient of viscosity and the thermal conductivity, and F_p and F_q are functions of these variables.

8. POWER SOLUTIONS

The Maxwellian is a unique solution of the equation

$$J = 0 \tag{8.1}$$

However, well-known conditions [4, 5] are imposed on the integrability of the distribution function in velocity space in this case. Otherwise it is possible to obtain power solutions [55].

The collision integral can be written in the form

$$J = \int w(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)F(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)d\tau$$

$$F(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3) = ff_1 - f_2f_3, \quad d\tau = d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3$$

Here $\mathbf{p}$ is the momentum, $E(\mathbf{p})$ is the collision energy of the particles, $w(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3) = U(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)\delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\delta(E + E_1 - E_2 - E_3)$ is the transition probability, $U(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)$ is the matrix element, and δ is the delta function. In this section we used the notation employed in [55]. The functions w and F possess the following properties

$$w(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3) = w(\mathbf{p}_1\mathbf{p}|\mathbf{p}_2\mathbf{p}_3) = w(\mathbf{p}_2\mathbf{p}_3|\mathbf{p}\mathbf{p}_1)$$

$$F(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3) = F(\mathbf{p}_1\mathbf{p}|\mathbf{p}_2\mathbf{p}_3) = -F(\mathbf{p}_2\mathbf{p}_3|\mathbf{p}\mathbf{p}_1)$$

Moreover, we assume that $E(\mathbf{p})$ and $U(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)$ are homogeneous functions

$$E(\lambda\mathbf{p}) = \lambda^{\beta}E(\mathbf{p}), \quad U(\lambda\mathbf{p}\lambda\mathbf{p}_1|\lambda\mathbf{p}_2\lambda\mathbf{p}_3) = \lambda^{m}U(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)$$

and the system is isotropic, i.e. the functions w and E are invariant under arbitrary rotations $\hat{g}$

$$E(\hat{g}\mathbf{p}) = E(\mathbf{p}), \quad w(\hat{g}\mathbf{p}\hat{g}\mathbf{p}_1|\hat{g}\mathbf{p}_2\hat{g}\mathbf{p}_3) = w(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)$$

We will seek a solution of Eq. (8.1) in the form

$$f \sim E^{S} \tag{8.2}$$

where s is a contant. Using the properties of symmetry of the collisions, and transformations of the rotation and extension, it is possible to reduce the collision integral to the form

$$J = \frac{1}{4}E^{\nu}\int w(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)F(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)[E^{-\nu} + E_1^{-\nu} - E_2^{-\nu} - E_3^{-\nu}]d\tau$$

When $F(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3)$ we obtain the Maxwell solution, and when the expression in square brackets vanishes we obtain the required solutions: $s = s_0$ when $\nu(s) = 0$ as a result of conservation of the number of particles, and $s = s_1$ when $\nu(s) = -1$ due to conservation of energy.

For particles with a power intermolecular potential $V(r) = V_0r^{-\alpha}$

$$s = \begin{cases} s_1 = -(m + 3d)/(2\beta), & m = -4 + 2\beta(1 - \alpha^{-1}) \\ s_0 = s_1 + 1/2 \end{cases}$$

Here d is the dimensionality of momentum space. For Maxwell molecules, when $\alpha = 4$ and $m = -1$, we have

$$s_0 = -3/2, \quad s_1 = -2 \tag{8.3}$$

According to relations (8.2) and (8.3), the distribution function $f \to \infty$ when $\mathbf{p} \to 0$, and the integral for the density (energy) of the particles either diverges at zero or at infinity, i.e. these solutions do not hold over the whole of velocity space. They are examples of intermediate asymptotic forms for open systems.

The search for such solution [55] was stimulated by research on the kinetic theory of weakly turbulent plasma [56], where the kinetic equation is formulated for the "density" of waves in wave-vector space.

Different examples of power solutions (in particular, for isotropic turbulence) were given in [55], which hold in regions in which self-similar solutions exist.

It should be noted that solutions which describe the occurrence of small flows in the region of equilibrium or stationary distributions are obviously realized more often than single-flow power solutions. Versions of approximate solutions, which differ only slightly from the power solutions described above were given in [55].

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